

A Local Version of the Poincaré-Hopf Theorem

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We describe an algebraic relationship which appears between a zero of a vector field X and a singular point of tangent hypersurface to X defined by f , being a local version of the Poincaré-Hopf index Theorem, in the isolated singularity case.

To the function f we associate its Jacobian algebra, which has a non-degenerate bilinear form, given by Grothendieck duality. Similarly, one may associate to X a finite dimensional algebra with a non-degenerate bilinear form, using the Jacobian determinant.

We will explain canonical decompositions that these algebras have, in the spirit of the Hard Lefschetz Theorem, where multiplication by f plays the role of "cutting with the hyperplane", that will give orthogonal decompositions of the bilinear forms.

Then using the tangency condition between the vector field and the hypersurface, we obtain relationships between some of the primitive pieces in the orthogonal decompositions of the corresponding algebras.

We will give formulas for the (GSV)index of the vector field restricted to the hypersurface, where the higher order terms in the formulas correspond to some of the above primitive pieces. This is specially revealing when working over the Real numbers, since then the bilinear forms have signatures as numerical invariants.

We will also mention extensions where the tangency is with a complete intersection.