Stand still or go away!

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The main problem:

Let X be an analytic vector-field defined on an analytic manifold M and let $N \subset M$ be a closed analytic sub-manifold. Suppose that N is **quasi-transversal** *to X, i.e. for every point $p \in N$, the dimension of $X(p) + T_pN$ is equal to the sum of the dimension of $X(p)$ and T_pN. Then, for every point $p \in N$, does it exist a $T = T(p) > 0$ and a neighborhood $U_p \subset N$ of p such that $exp(tX)(U_p - Sing(X))$ don't intersect N for $0 < |t| < T(p)$?*

Some examples:

If $p \in N \setminus Sing(X)$ then the result is trivial near p: this is a consequence of the flow-box theorem. But if $p \in Sing(X)$ the problem becomes non-trivial as illustrates the figure:

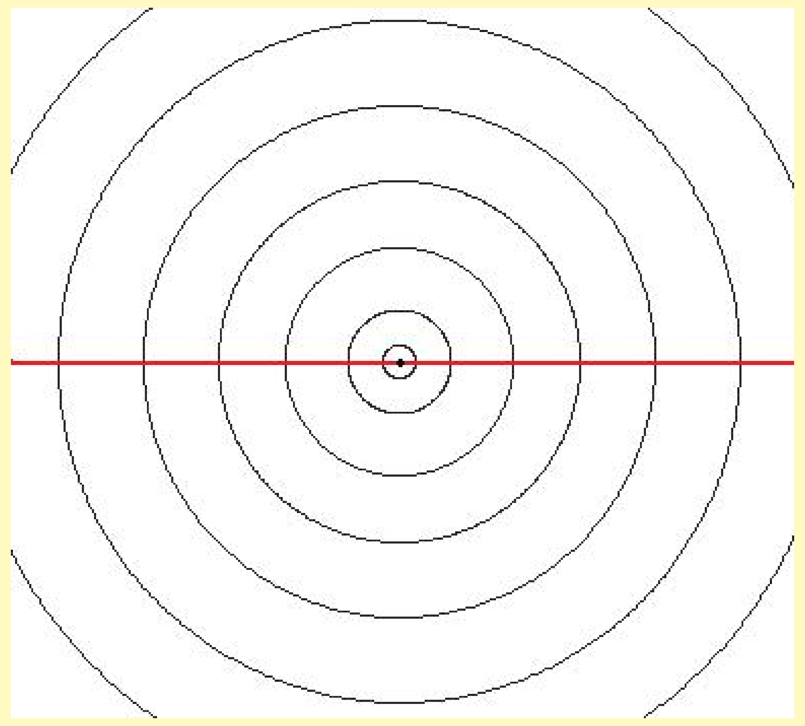

Fig: In the figure N is the red curve and X is a center type vector-field

A more general problem:

One can formulate a more general problem:

Let θ be an involutive singular distribution defined on an analytic manifold M and let N be a closed analytic sub-manifold of M. Fix a sub-Riemannian metric $g : T\theta \otimes T\theta \to \mathbb{R}$ and let d be the sub-Riemannian distance associated to g. Then, for every point $p \in N$, does it exist a $T = T(p) > 0$ and a neighborhood $U_p \subset N$ such that $d(q, N \setminus q) > T$ for all $q \in U_p$?

Motivation

This problem is motivated by a question of Jean-Fraçois Mattei, concerning the existence of sections for the action of a Lie group on an analytic manifold.

Main Definitions: Transversality

A vector-field X and a sub-manifold N are:

- *geometrically quasi-transversal* if for every point $p \in N$, the dimension of $X(p) + T_pN$ is equal to the sum of the dimension of $X(p)$ and T_pN;
- *algebraically quasi-transversal* if they are geometrically quasi-transversal and $X(X(\mathcal{I}_N)) \subset < X(\mathcal{I}_N) + \mathcal{I}_N >$, where $< S >$ stands for the ideal generated by S and $\mathcal{I}_N$ is the radical ideal sheaf whose support is N.

There exists a more general definition for the case of a singular distribution.

First Remark: Invariant Blow-ups

A blow-up with smooth center:

$$\sigma : M^{'} \to M$$

is called *invariant* if the center $\mathcal{C}$ is invariant by X, i.e. all orbits of X that intersect $\mathcal{C}$ are all contained in $\mathcal{C}$. In this case the pull-back X' of X is analytic and the "time" along the orbits of X and X' is the same (outside the exceptional divisor). No other kind of blow-up has this property. We denote by N' the strict transform of N under σ.

We remark that no point in the exceptional divisor is in the transform $\sigma^{-1}(N \setminus Sing(X))$.

Second Remark: Local reduction

If X and N are geometrically quasi-transversal (respectively algebraically) and

$$\sigma : M' \to M$$

is an invariant blow-up then X' and N' are geometrically quasi-transversal (respectively algebraically) on $M' \setminus \sigma^{-1}(C)$.

If the main problem has positive answer for all points of $N^{'}$ then it has positive answer for N because of the properness of σ. Furthermore, remark that orbits in the exceptional divisor are blown-down to $Sing(X)$.

Blow-up reduction:

Theorem: *For any $M_0 \subset M$ a relatively compact subset, there exists a sequence of invariant blow-ups:*

$$M_r \overset{\sigma_r}{\to} ... \overset{\sigma_2}{\to} M_1 \overset{\sigma_1}{\to} M_0$$

such that:

- *If X and N are algebraically quasi-transversal, then X_r is everywhere transversal to N_r;*
- *If X and N are geometrically quasi-transversal, then for each $p \in N_r$, X_r is either transversal or finitely tangent.*

This theorem, as it is enunciated, can not be generalized for an arbitrary singular distribution.

Resolved cases:

Corollary: *If X and N are algebraically quasi-transversal, then the main problem has positive answer.*

Corollary: *If N has dimension 1 or codimension 1, the main problem has positive answer. In particular, the problem is solved for M of dimension ≤ 3.*

These two results are also valid for an involutive singular distribution θ.

In the geometrically quasi-transversal case, the singularity is hiding tangencies. But this does not mean the existence of a counter-example.

Below, we illustrate this phenomena.

An example of the remaining difficulty:

The difficulties appear only in dimension ≥ 4: Let $M = \mathbb{R}^4$ and $N = V((z, w))$ and

$$X = x\frac{\partial}{\partial z} + y^2\frac{\partial}{\partial w} + y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$$

It is clear that X and N are geometrically quasi-transversal but not algebraically quasi-transversal. After a blow-up on the origin, consider the y-chart. In this chart $N^{'} = V((z, w))$ and:

$$X^{'} = (x - xz)\frac{\partial}{\partial z} + (y - xw)\frac{\partial}{\partial w} + (1 - x^2)\frac{\partial}{\partial x} - xy\frac{\partial}{\partial y}$$

which is tangent to $N^{'}$ on the origin. But this is not a counter-example just as we see in the figures (and there is a very simple proof of that).

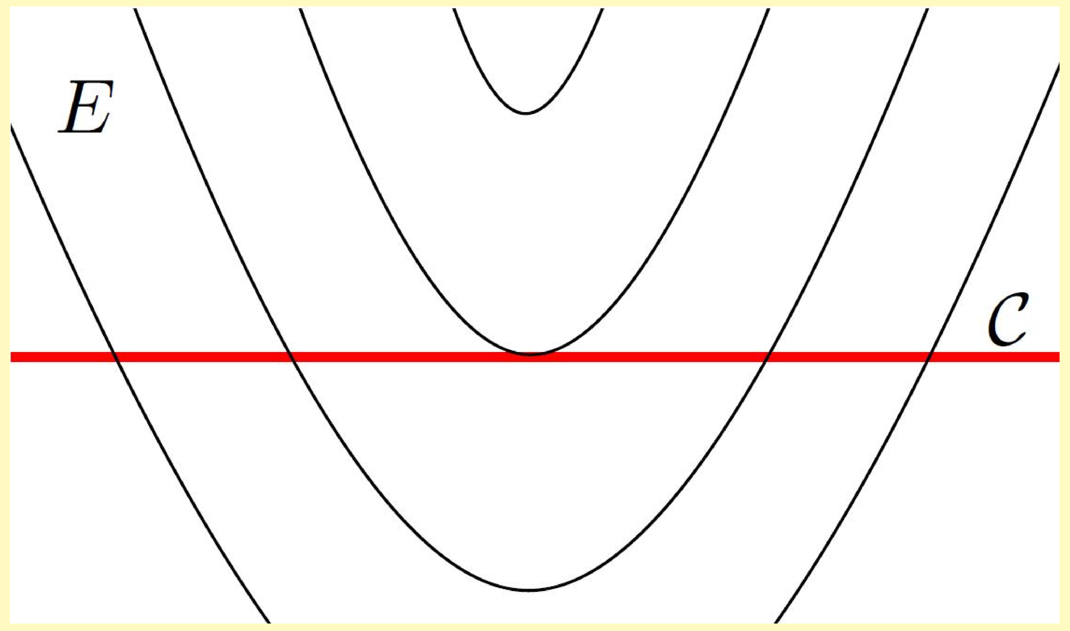

Fig1: We restrict everything to the exceptional divisor E. Notice that $X^{'}$ is tangent to $N^{'}$.

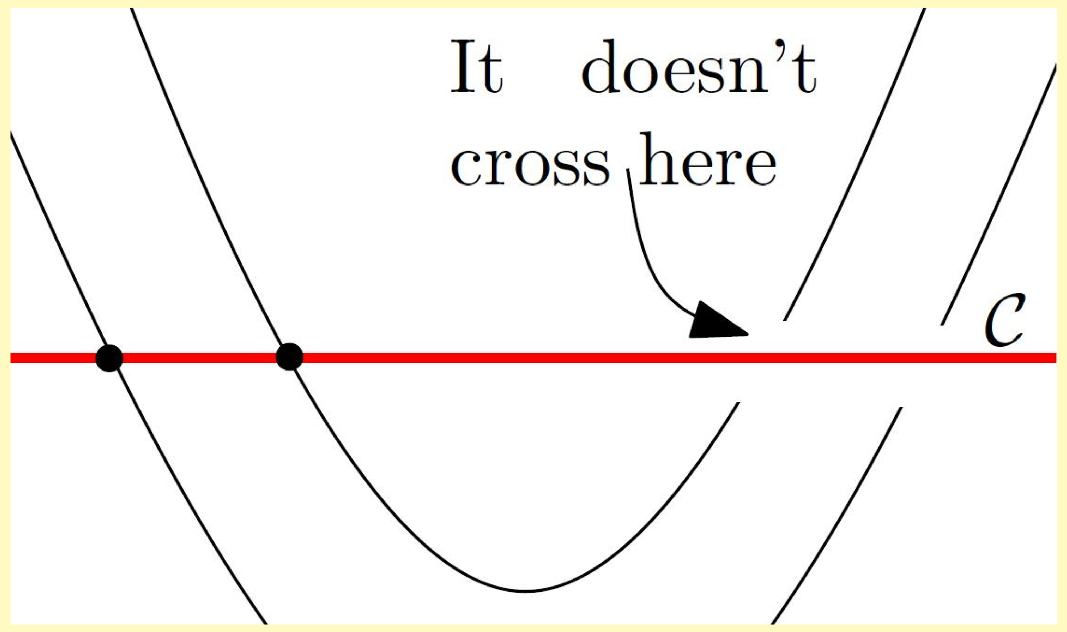

Fig2: We restrict everything to a three-manifold outside the exceptional divisor E. Notice that the orbits of $X^{'}$ intersects $N^{'}$ only one time.

For a more challenging example, consider $N = V((z, w))$ and

$$X = x\frac{\partial}{\partial z} + (y^2 + z)\frac{\partial}{\partial w} + y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$$