A NEW EXAMPLE OF HILBERT MODULAR FOLIATIONS

GAËL COUSIN

1. Hilbert modular foliations

1.1. **The Group.** Let $K = \mathbb{Q}(\sqrt{d})$, for a positive squarefree integer d. Let $\mathcal{O}_K$ be the ring of integers of K.

The field K embeds in two manners in $\mathbb{R}$, which yields two embeddings $(\Psi_i)_{i=1,2}$ of $\mathrm{PSL}_2(\mathcal{O}_K)$ in $\mathrm{PSL}_2(\mathbb{R})$. We get

$$\begin{array}{rccc} \Psi : & \mathrm{PSL}_2(\mathcal{O}_K) & \hookrightarrow & \mathrm{PSL}_2(\mathbb{R}) \times \mathrm{PSL}_2(\mathbb{R}) \\ & A & \longmapsto & (\Psi_1(A), \Psi_2(A)) \end{array}$$

The image of that morphism is a discrete subgroup of $\mathrm{PSL}_2(\mathbb{R})^2$. We call it $\Gamma_{\sqrt{d}}$.

1.2. **The Surface.** The quotient $\mathbb{H} \times \mathbb{H}/\Gamma_{\sqrt{d}}$ has a natural quasiprojective singular surface structure : it becomes compact-projective after adding a finite number of points ("cusps"). The resulting compact surface has a natural desingularisation we call $Y_{\sqrt{d}}$.

1.3. **Its Natural Foliations.** The complex surface $Y_{\sqrt{d}}$ is endowed with two foliations, which are the projections of the natural foliations which arise from the product structure of $\mathbb{H} \times \mathbb{H}$. We call them Hilbert modular foliations. They are denoted $\mathcal{F}_{\sqrt{d}}$ and $\mathcal{G}_{\sqrt{d}}$.

1.4. **An Involution.** The natural involution which permutes the factors of $\mathbb{H} \times \mathbb{H}$ yields a holomorphic involution of $Y_{\sqrt{d}}$ which permutes $\mathcal{F}_{\sqrt{d}}$ and $\mathcal{G}_{\sqrt{d}}$.

1.5. **Rationality of the Surface.** A natural question about $Y_{\sqrt{d}}$ is to know its position in Enriques-Kodaira classification of surfaces. The answer to that question was given for every d by Hirzebruch, Van de Ven and Zagier in a series of papers.

A part of the result is that $Y_{\sqrt{d}}$ is rational if and only if

$$d = 2, 3, 5, 6, 7, 13, 17, 15, 21 \text{ or } 33.$$

1.6. **Known Example.** In their paper [MP05] Mendes and Pereira give, among other things, a birational model on $\mathbb{P}^2$ for the pair $(\mathcal{F}_{\sqrt{5}}, \mathcal{G}_{\sqrt{5}})$. They obtain these models thanks to detailed informations on $Y_{\sqrt{5}}$ given by Hirzebruch in [Hir76].

2. Results

With a new method, we obtain a model for the pair $(\mathcal{F}_{\sqrt{3}}, \mathcal{G}_{\sqrt{3}})$.

This the work of [Cou12].

Theorem. *A birational model for the twice foliated surface* $(Y_{\sqrt{3}}, \mathcal{F}_{\sqrt{3}}, \mathcal{G}_{\sqrt{3}})$ *is* $(\mathbb{P}^2, \mathcal{F}_\omega, \mathcal{G}_\tau)$ *where*

$$\omega = 6\,\left(3\,v^2+1\right)v\left(v^2+9\,uv^2+3\,u\right)du \\ +\left((9\,u-5)\,(9\,u-2)\,(9\,u-1)\,v^4+9\,u\left(5+54\,u^2-30\,u\right)v^2+9\,u^2\,(9\,u-2)\right)dv.$$

and

$$\tau = 6\,\left(3\,v^2+1\right)v\left(-8\,v^2-3+36\,uv^2+12\,u\right)du \\ +\left((9\,u-5)\,(9\,u+1)\,(9\,u-1)\,v^4+\left(3+486\,u^3-432\,u^2+45\,u\right)v^2+9\,u\,(9\,u-2)\,(u-1)\right)dv.$$

Moreover, $\sigma : (u, v) \mapsto \left(\frac{3\,v^2\left(36\,v^2+13\right)u-v^2\left(20\,v^2+9\right)}{9\,(12\,v^2-1)(3\,v^2+1)u-3\,v^2(36\,v^2+13)}, v\right)$ *is a birational involution of* $\mathbb{P}^2$ *which permutes* $\mathcal{F}_\omega$ *and* $\mathcal{G}_\tau$*.*

The equations for this example are obtained from the transversely projective structures of the foliations. We use classification results for foliations on surfaces due to Brunella, McQuillan and Mendes to prove these equations indeed yield the required foliations.

3. Transversely projective foliations

3.1. **Transversely Projective Foliations.** Let $\mathcal{H}$ be a codimension 1 foliation on a complex manifold M. A *transversely projective structure* for $\mathcal{H}$ is the data of a triple $(\pi, \mathcal{R}, \sigma)$ formed by

(1) a $\mathbb{P}^1$-bundle $\pi : P \to M$;
(2) a codimension 1 holomorphic singular foliation $\mathcal{R}$ on P which is transverse to π over $M \setminus D$, for an hypersurface D on M.
(3) a meromorphic section σ of π, such that $\mathcal{H} = \sigma^*\mathcal{R}$.

When we have only (1) and (2), we say $\mathcal{R}$ is a Riccati foliation on π.

3.2. **Monodromy.** In the above situation, if $\star \in M \setminus D$, lifting looks in $M \setminus D$ to th leaves of $\mathcal{R}$, we can define a morphism

$$\pi_1(M \setminus D, \star) \to Aut(\pi^{-1}(\star)).$$

This morphism is called the monodromy representation of $\mathcal{R}$.
Choosing an isomorphism $Aut(\pi^{-1}(\star)) \simeq \mathrm{PSL}_2(\mathbb{C})$ we can describe this representation thanks to matrices.

3.3. **The case of Hilbert Modular Foliations.** The foliations $\mathcal{F}_{\sqrt{d}}$ and $\mathcal{G}_{\sqrt{d}}$ have transversely projective structures. The images of their monodromy representations are both $\mathrm{PSL}_2(\mathcal{O}_K) \subset \mathrm{PSL}_2(\mathbb{C})$.

4. Construction of our example

Well Chosen Algebraic Solution to Painlevé VI Equation

↓ general formulae

Riccati foliation $(\pi, \mathcal{R})$ over $\mathbb{P}^2$

↓ quotient under an involution

Riccati foliation $(\tilde{\pi}, \tilde{\mathcal{R}})$ over $\mathbb{P}^2$

↓ choice of a section σ of $\tilde{\pi}$

Transversely projective foliation $\mathcal{F} = \sigma^* \tilde{\mathcal{R}}$.

5. Proof of Modularity

The proof that the foliation $\mathcal{F}$ described above is a modular foliation relies on the classification results for foliations on surfaces *à la* Enriques-Kodaira due to Brunella, McQuillan and Mendes, see [Bru00] and references therein. For the completion of this classification see [Bru03]. It also relies on the theory of transversely projective foliations developed by Scárdua [Scá97], Cerveau-Lins-Neto-Loray-Pereira-Touzet[CLNL$^+$07], and Loray-Pereira [LP07].

In that manner we obtain the proposition mentioned below. Then the study of the transverse structure of $\mathcal{F}$ and its monodromy permits to use the key-proposition, the result of which, combined with the properties of the pair $(\mathcal{F}_{\sqrt{d}}, \mathcal{G}_{\sqrt{d}})$, allows to obtain the theorem mentioned above.

Key-Proposition. *Let $\mathcal{F}$ be a reduced foliation on a smooth projective surface.*

If

- $\nu(\mathcal{F}) = 1$,
- *$\mathcal{F}$ is a transversely projective folaition $\mathcal{F} = \sigma^*\mathcal{R}$,*
- *$\mathcal{R}$ is not pull-back of a Riccati foliation over a curve,*
- *the monodromy of $\mathcal{R}$ is not virtually abelian;*

then $\mathcal{F}$ is a modular foliation.

References

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