Classification of diffeomorphisms and ε-neighborhoods of orbits

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1. Fractal properties of a set

$U_\varepsilon \ldots \varepsilon$-neighbourhood of a measurable set $U \subset \mathbb{R}^N$

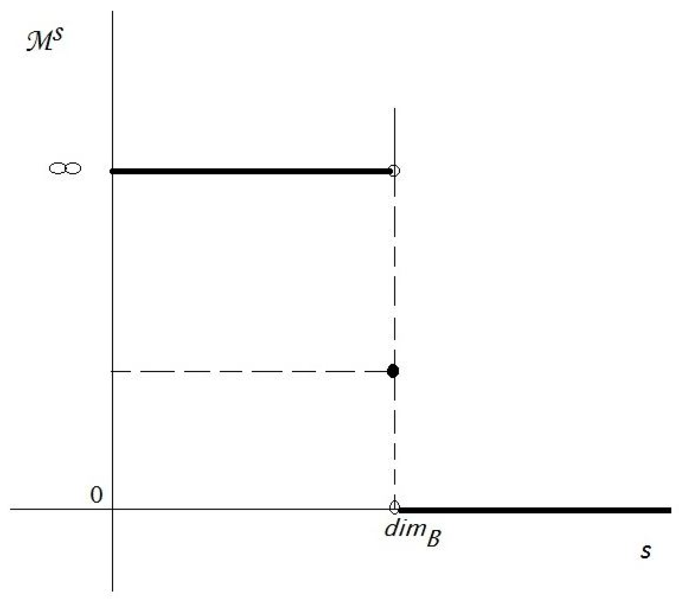
$A(U_\varepsilon) \ldots$ its Lebesgue measure

- *The Minkowski content of* U, $0 \le s \le N$:

$$\mathcal{M}^s(U) = \lim_{\varepsilon \to 0} \frac{A(U_\varepsilon)}{\varepsilon^{N-s}}$$

- *The box dimension* of U: $\overline{\dim}_B U = \inf\{s \ge 0 \mid \mathcal{M}^s(U) = 0\}$.

* $A(U_\varepsilon) \simeq C\varepsilon^K,\ \varepsilon \to 0 \Rightarrow d = \dim_B(U) = N - K,\ \mathcal{M}^d(U) = C.$



The function $s \mapsto \mathcal{M}^s(U)$

2. Can one read formal classification of a parabolic diffeomorphism from ε-neighborhood of its orbit? [R]

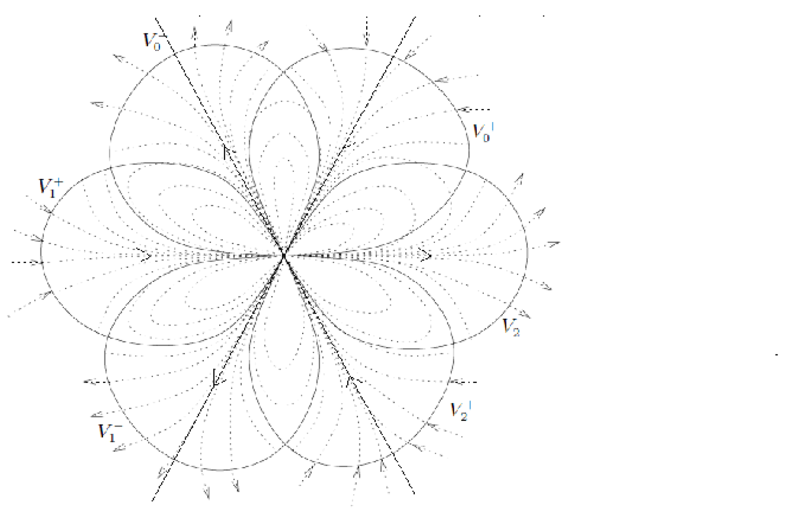
$\star$ **Yes, and we need only first $k+1$ terms in the development!**

$f \in Diff(\mathbb{C}, 0)$ a *germ of parabolic diffeomorphism*,

$$(*)\ f(z) = z + a_1 z^{k+1} + a_2 z^{k+2} + \ldots + a_k z^{2k+1} + o(z^{2k+1}),\ a_i \in \mathbb{C},\ a_1 \neq 0.$$

Discrete orbits near the origin – ***Leau-Fatou flower***:

k attracting directions: $(-a_1)^{-\frac{1}{k}}$, k repelling directions: $a_1^{-\frac{1}{k}}$



Attracting/repelling directions and sectors for $f(z) = z + z^4 + o(z^4)$

<ins>Formal classification of parabolic diffeomorphisms</ins>

Formal changes of variables **tangent to the identity**

$\Rightarrow (*)$ reduces to its *expanded formal normal form*

$$g(z) = z + a_1 z^{k+1} + a z^{2k+1},\ a \in \mathbb{C}.$$

<ins>The development of ε-neighborhood, $\varepsilon \to 0$:</ins>

$*$ The **directed area** of the ε-neighborhood of an orbit $S^f(z_0)$ (z_0-initial point):

$$A^{\mathbb{C}}(S^f(z_0)_\varepsilon) = A(S^f(z_0)_\varepsilon) \cdot \nu_{\mathbf{t}(S^f(z_0)_\varepsilon)} \in \mathbb{C},$$

$\nu_{\mathbf{t}(S^f(z_0)_\varepsilon)}$ the normalized center of mass of $S^f(z_0)_\varepsilon$, $A(S^f(z_0)_\varepsilon)$ the area of $S^f(z_0)_\varepsilon$.

$$\Rightarrow\ A^{\mathbb{C}}(S^f(z_0)_\varepsilon) = \mathbf{K_1}\varepsilon^{\mathbf{1+\frac{1}{k+1}}} + K_2\varepsilon^{1+\frac{2}{k+1}} + \ldots + K_{k-1}\varepsilon^{1+\frac{k-1}{k+1}} +$$
$$+ K_k\varepsilon^{1+\frac{k}{k+1}}\log\varepsilon + K_k'\varepsilon^{1+\frac{k}{k+1}} + \mathbf{K_{k+1}}\varepsilon^2\log\varepsilon + o(\varepsilon^2\log\varepsilon),\ \varepsilon \to 0,\ K_i \in \mathbb{C},\ (1)$$

Coefficients in the development $\leftrightarrow$ fractal properties of the orbit:

- $\dim_B(S^f(z_0)) = 1 - \frac{1}{k+1}$
- The **complex Minkowski content** of $S^f(z_0)$: $\mathcal{M}^{\mathbb{C}}(S^f(z_0)) = \lim_{\varepsilon\to 0} \frac{A^{\mathbb{C}}(S^f(z_0)_\varepsilon)}{\varepsilon^{1+\frac{1}{k+1}}} = K_1 \in \mathbb{C}$.
- The **residual content of** $S^f(z_0)$ – the coefficient in front of $\varepsilon^2\log\varepsilon$, $\mathcal{R}^{\mathbb{C}}(S^f(z_0)) = K_{k+1} \in \mathbb{C}$.

<ins>**Theorem**</ins> **1 (R)** *There exists a **bijective correspondence** between the triple*

$$\Big(\dim_B(S^f(z_0)),\ \mathcal{M}^{\mathbb{C}}(S^f(z_0)),\ \mathcal{R}^{\mathbb{C}}(S^f(z_0))\Big),\ \textit{that is,}\ \Big(k,\ K_1,\ K_{k+1}\Big),$$

and the formal type of f,

$$(k,\ a_1,\ a).$$

*Here, $S^f(z_0)$ is **any** attracting orbit of f.*

3. What can be said about the analytic classification of parabolic diffeomorphism from ε-neighborhoods of its orbits?

* $f(z)$ analitically conjugate to its formal normal form **only sectorially**

* analytic classification: FNF $+\ 2k$ diffeomorphisms relating sectors (*Voronin's moduli of analytic classification*).

Is it sufficient to study directed area of the ε-neighborhood of **just one orbit**?

<ins>**Theorem**</ins> **2** *The germ $f(z)$ (and then also its analytic class) is uniquely determined by the pair*

$$\big(A(S^f(z_0)_\varepsilon),\ \mathbf{t}(S^f(z_0))_\varepsilon\big),\ \varepsilon < \varepsilon_0,$$

where $S^f(z_0)$ is any attracting orbit!

Open problem:

Can we replace $\big(A(S^f(z_0)_\varepsilon),\ \mathbf{t}(S^f(z_0))_\varepsilon\big)$ by $A^{\mathbb{C}}(S^f(z_0)_\varepsilon)$? Is any information about diffeomorphism lost normalizing the center of mass?

Can we obtain **analytic classification complexifying** $A^{\mathbb{C}}(S^f(z_0)_\varepsilon)$ by ε?

Can we apply the Borel-Laplace transform to its formal asymptotic development, as $\varepsilon \to 0$?

Problems:

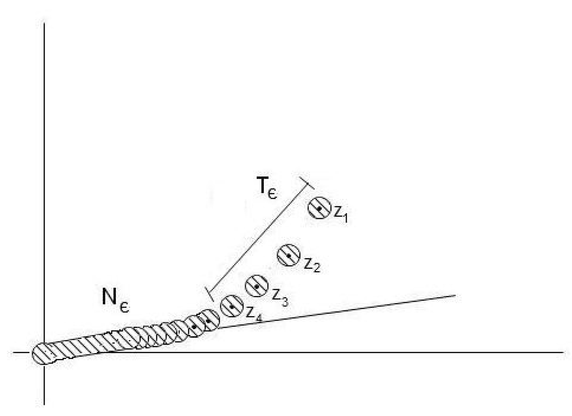
(1) $A(S^f(z_0))_\varepsilon$ as function of real $\varepsilon > 0$ has **accumulation of singularities at** $\varepsilon = 0$!

<ins>**Theorem**</ins> **3** *The function $A(S^f(z_0))_\varepsilon$ is a C^1 function on some interval $(0, \varepsilon_0)$, but not even of the class C^2. There exists a decreasing sequence $\varepsilon_n \to 0$, $n \to \infty$, such that*

$$\lim_{\varepsilon \to \varepsilon_n+} \frac{d^2}{d\varepsilon^2} \frac{A(S^f(z_0)_\varepsilon)}{\varepsilon^2\pi} = -\infty.$$

On each open subinterval $(\varepsilon_{n+1}, \varepsilon_n)$, $n \in \mathbb{N}$, $A(S^f(z_0))_\varepsilon$ is analytic.

– singularities: $\varepsilon_n = \frac{|z_{n+1} - z_n|}{2}$



The area of the ε-neighborhood – area of the tail and the nucleus:

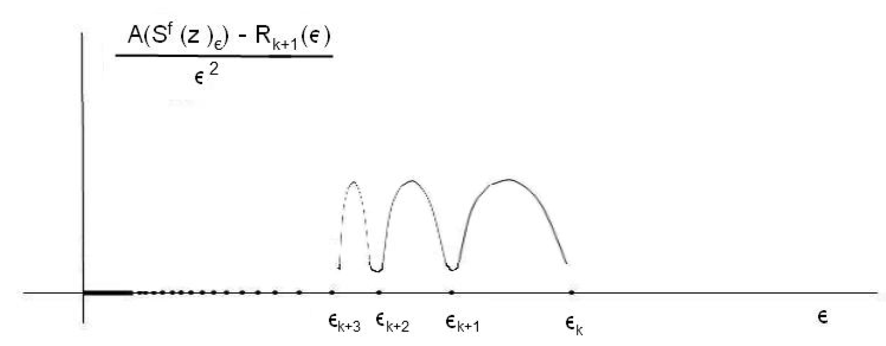
$A(S^f(z_0)_\varepsilon) = A(T_\varepsilon)$ (tail!) $+\ A(N_\varepsilon)$ (nucleus!)

(2) $A(S^f(z_0)_\varepsilon)$ has **asymptotic development only up to first** $(k+1)$ **terms**!

$$A(S^f(z_0)_\varepsilon) = R_{k+1}(\varepsilon) + O(\varepsilon^2),$$

$$\limsup_{\varepsilon\to 0} \frac{A(S^f(z_0)_\varepsilon) - R_{k+1}(\varepsilon)}{\varepsilon^2} \neq \liminf_{\varepsilon\to 0} \frac{A(S^f(z_0)_\varepsilon) - R_{k+1}(\varepsilon)}{\varepsilon^2},$$

where $R_{k+1}(\varepsilon)$ are first $k+1$ terms in the development (1).



$\Rightarrow A(S^f(z_0))_\varepsilon$ does not have an asymptotic development as $\varepsilon \to 0$ in the given power-logarithmic scale, **unable to perform standard resummation procedure ???**

<ins>References</ins>:

[R] Resman, M., *ε-neighborhoods of orbits and formal classification of parabolic diffeomorphisms*, to be published in Discrete and Continuous Dynamical Systems, Series A, ArXiv: 1207.2954v2