Confluence of q-difference equations in the irregular singular case.

Dreyfus Thomas

Institut de mathématiques de Jussieu, Topologie et géométrie algébriques.

tdreyfus@math.jussieu.fr

Abstract

We study formal power series solution of q difference equations. The series may diverges but as in the differential case, there exists an asymptotic solution. We see q as a parameter, and we make q goes to 1. Under convenient assumptions, we prove that the q-asymptotic solution converge to the asymptotic solution of a differential equation. We treat the example of the Euler equation.

Introduction

Let us fix $q \in]1,\infty[$ and let us define $\sigma_q f(X) = f(qX)$. The operator $\delta_q = \frac{\sigma_q - \mathrm{Id}}{q-1}$ formally converges to the differential operator $\delta = X\frac{d}{dX}$. It will be natural to have the convergence of the solutions of the δ_q-system $\delta_q Y(X,q) = B(X,q)Y(X,q)$ to the solutions of the δ-system $\delta \widetilde{Y}(X) = \widetilde{B}(X)\widetilde{Y}(X)$, where $X \mapsto B(X,q) \in \mathrm{GL_m}(\mathbb{C}(X))$ converges to $\widetilde{B}(X,q) \in \mathrm{GL_m}(\mathbb{C}(X))$ under convenient assumptions.
This problem has been treated by J. Sauloy in [S00], when the systems are Fuchsian (i.e : then matrices has no poles in $X = 0$).
We present here a particular case of a theorem present in [D]. Let us consider a formal power series $\widetilde{f} \in \mathbb{C}[[X]]$, solution of a linear differential equation of order 1 and rank m in coefficient in $\mathbb{C}(\{X\})$, the field of germs of meromorphic function. In general, the serie is divergent, but there exists one and only one function f, that is solution of the same equation, that is a germ of analytic function in some sector $\{X \in \mathbb{C}^* | \arg(X) \in]a,b[\}$, and that is Gevrey asymptotic to $\hat{f}$ (see [R79, VdPS]). The asymptotic solution can be computed using Borel and Laplace transformations.
The same type of phenomenon occurs when we consider formal series solution of δ_q-equations of order 1 and rank m and it has been studied in [RSZ]. This time, the analytic sum will be meromorphic on $\mathbb{C}^*$, with poles on some q-spiral $-\lambda q^{\mathbb{Z}}$. When the slopes are 0 and 1, we are able to compute the q-sum using q-Laplace and q-Borel transformations. See [DVZ].
We will consider family of δ_q-equation that converges to a δ-equation. Under convenient assumptions, we will see that the q-sum converges to the differential sum.
The poster is presented as follow. In the first two parts, we make a review of the existance of assymptotic solution for the differential and the q-difference equations. In the last part, we study the convergence of the q-sum to the differential sum.

Asymptotic solutions of differential equations

Let us consider a formal power serie $\widetilde{f}(X) \in X\mathbb{C}[[X]]$, solution of the δ-equation :

$$\widetilde{a}_m(X)\delta^m\widetilde{y}(X) + \widetilde{a}_{m-1}(X)\delta^{m-1}\widetilde{y}(X) + \cdots + \widetilde{a}_0(X)\widetilde{y}(X) = 0, \quad (1)$$

with $\widetilde{a}_m\widetilde{a}_0 \neq 0$ and $\widetilde{a}_k(X) \in \mathbb{C}(\{X\})$.
The Newton polygon of this equation is the convex hull of the $\{(i,j) \in \mathbb{N}^* \times \mathbb{Z} | i \leq k, j \geq v_0(\widetilde{a}_k)\}$, where v_0 denotes the X-adic valuation. Let $(m_1, d_1), \dots, (m_k, d_k)$ chosen minimal with $m_1 < \cdots < m_k$ such that the lower part of the boundary of the Newton polygon of this equation is the convex hull of $(m_1, d_1), \dots, (m_k, d_k)$. We call slopes of the δ-equation the $\mu_i = \frac{d_{i+1} - d_i}{m_{i+1} - m_i} \in \mathbb{Q}$. We will assume that the slopes are 0 and 1.
There exists a finite subset $S\left(\widetilde{f}\right) \subset [0, 2\pi[$, such that for all $d \in [0, 2\pi[\backslash S\left(\widetilde{f}\right)$, there exists one and only one solution of the equation (1) $\widetilde{f}^d$, which is a germ of meromorphic function on the sector $\left\{X \in \mathbb{C}^* | \arg(X) \in \left]d - \frac{\pi}{2}, d + \frac{\pi}{2}\right[\right\}$ and Gevrey asymptotic to $\widetilde{f}$.
We can compute the $\widetilde{f}^d$ using Laplace and Borel transformations. We recall the definition now.

Definition 1 – *The formal Borel transforms is defined as follows :*

$$\hat{\mathcal{B}}\left(\sum a_n X^{n+1}\right) = \sum \frac{a_n}{n!} X^n.$$

– The Laplace transform of f is defined as follows :

$$\mathcal{L}^d(f(X)) = \int_0^{\infty e^{id}} f(u) e^{-\frac{u}{X}} du.$$

For all $d \in [0, 2\pi[\backslash S\left(\widetilde{f}\right)$, we have the relation :

$$\widetilde{f}^d = \mathcal{L}^d \circ \hat{\mathcal{B}}\left(\widetilde{f}\right).$$

Example 2 *Let us treat now the example of the Euler equation :*

$$\delta^2\widetilde{Y}(X) = -\frac{\delta\widetilde{Y}(X)}{X} + \frac{\widetilde{Y}(X)}{X}.$$

$\widetilde{f}(X) = \sum (-1)^n n! X^{n+1}$ *is a formal solution. For d not congruent to π mod 2π :*

$$\widetilde{f}^d(X) = \int_0^{\infty e^{id}} \frac{1}{(1+\zeta)e\left(\frac{\zeta}{X}\right)} d\zeta.$$

Asymptotic solution of q-difference equations

In this part, $q > 1$ will be a fixed real number. Let us consider a formal power serie $\hat{f}(X) \in X\mathbb{C}[[X]]$, solution of the δ_q-equation :

$$a_m(X)\delta_q^m y(X) + a_{m-1}(X)\delta_q^{m-1} y(X) + \cdots + a_0(X)y(X) = 0, \quad (2)$$

with $a_m a_0 \neq 0$ and $a_k(X) \in \mathbb{C}(\{X\})$.
The δ_q-equation can be seen as a σ_q-equation :

$$b_m(X)\sigma_q^m y(X) + b_{m-1}(X)\sigma_q^{m-1} y(X) + \cdots + b_0(X)y(X) = 0.$$

The Newton polygon of the σ_q-equation is the convex hull of $\{(i,j) \in \mathbb{N}^* \times \mathbb{Z} | j \geq v_0(b_i)\}$. Let $(m_1, d_1), \dots, (m_k, d_k)$ chosen minimal with $m_1 < \cdots < m_k$ such that the lower part of the boundary of the Newton polygon of this equation is the convex hull of $(m_1, d_1), \dots, (m_k, d_k)$. We call slopes of the σ_q-equation the $\mu_i = \frac{d_{i+1} - d_i}{m_{i+1} - m_i} \in \mathbb{Q}$. We will assume that the slopes of the σ_q-equation are 0 and 1.
There exists a finite subset $S\left(\hat{f}\right) \subset \mathbb{C}^*/q^{\mathbb{Z}}$, such that for all $\lambda \in \left(\mathbb{C}^*/q^{\mathbb{Z}}\right) \backslash S\left(\hat{f}\right)$, there exists one and only one solution of the equation (2) $\hat{f}^{[\lambda]}$, which is a germ of meromorphic function with simple poles on the q-spiral $-\lambda q^{\mathbb{Z}}$ and q-Gevrey asymptotic to $\hat{f}$.
As in the differential case, we can compute the $\hat{f}^{[\lambda]}$ using Laplace and Borel transformations. Let $[n]_q^! = \prod_{i \leq n} \frac{q^i - 1}{q-1}$ and let $e_q(X) = \sum \frac{X^n}{[n]_q^!}$.

Definition 3 – *We define the q-Borel transforms as follow :*

$$\hat{\mathcal{B}}_q\left(\sum a_n X^{n+1}\right) = \sum \frac{a_n}{[n]_q^!} \zeta^n$$

– Let $p = q^{-1}$. We define the q-Laplace transforms as follows

$$\mathcal{L}_q^{[\lambda]}(f)(X) = \frac{q}{1-p} \int_{\lambda p^{\mathbb{Z}}} \frac{f(\zeta/(1-p))}{e_q\left(\frac{q\zeta}{(1-p)X}\right)} d_p\zeta,$$

where

$$\int_{\lambda p^{\mathbb{Z}}} f(t) d_p t = (1-p)\lambda \sum_{l \in \mathbb{Z}} f(p^l \lambda) p^l.$$

For all $\lambda \in \left(\mathbb{C}^*/q^{\mathbb{Z}}\right) \setminus S\left(\hat{f}\right)$, we have the relation :

$$\hat{f}^{[\lambda]} = \mathcal{L}_q^{[\lambda]} \circ \hat{\mathcal{B}}_q\left(\hat{f}\right).$$

Example 4 *Let us treat the example of the q-deformation of the Euler equation.*

$$\delta_q^2 \hat{y}(X,q) - \frac{\delta_q \hat{y}(X,q)}{qX} + \frac{\hat{y}(X)}{qX} = 0.$$

The formal serie $\hat{f}(X,q) = \sum_{n \geq 0} (-1)^n [n]_q^! X^{n+1}$ *is solution. Let* $\lambda \in \left(\mathbb{C}^*/q^{\mathbb{Z}}\right) \setminus \{-1\}$. *The q-asymptotic solution is*

$$\hat{f}^{[\lambda]} = \frac{q}{1-p} \int_{\lambda p^{\mathbb{Z}}} \frac{g(\zeta/(1-p))}{e_q\left(\frac{q\zeta}{(1-p)X}\right)} d_p\zeta,$$

where $g(\zeta) = \frac{1}{1+\zeta}$

Confluence of the q-sum to the differential sum

In this part, we consider $q > 1$ as a parameter. We will make q goes to 1. Let us consider the family of equations :

$$a_m(X)\delta_q^m \hat{f}(X,q) + \cdots + a_0(X)\hat{f}(X,q) = 0, \quad (3)$$

$$a_m(X)\delta^m \widetilde{f}(X) + \cdots + a_0(X)\widetilde{f}(X) = 0, \quad (4)$$

with $a_m a_0 \neq 0$, $a_k(X) \in \mathbb{C}(\{X\})$, $\widetilde{f}(X), X \mapsto \hat{f}(X,q) \in \mathbb{C}[[X]]$ and with slopes 0 and 1. We will assume that the X-coefficients of $\hat{f}(X,q)$ are uniformly continuous in q.

Lemma 5 *Let $d \notin S\left(\widetilde{f}\right)$. Then for q sufficiently close to 1, $e^{id} \notin S\left(\hat{f}(X,q)\right)$.*

Theorem 6 (D) *Let $d \notin S\left(\widetilde{f}\right)$. Let $\widetilde{\Omega}$ be the domain of definition of $\widetilde{f}^d$. Then in every compact subset of $\widetilde{\Omega}$, $\hat{f}^{[e^{id}]}$ converges uniformly to $\widetilde{f}^d$ as q tends to 1.*

Example 7 *The theorem can be applied for the Euler equation.*

$$\delta_q^2 \hat{y}(X,q) - \frac{\delta_q \hat{y}(X,q)}{qX} + \frac{\hat{y}(X)}{qX} = 0, \quad (5)$$

$$\delta^2 \widetilde{Y}(X) + \frac{\delta\widetilde{Y}(X)}{X} - \frac{\widetilde{Y}(X)}{X} = 0. \quad (6)$$

Notice that this example has been already treated in [DVZ].

Conclusion and further expectations.

We give here a confluence result which improve the one in [DVZ]. We wish now to make the result more explicit in order to give an algorithm that compute a numerical approximation of the sum and the Stokes operators. In the differential case, in order to compute the Laplace transformation, we have to compute the analytic continuation of the Borel transformations along the path from 0 to ∞e^{id}. In the q-difference case, we have to compute the value of the analytic continuation of the q-Borel transformations along the q-spiral $\lambda q^{\mathbb{Z}}$. This problem is a priori more simple, because the q-Borel transformation satisfies a q-difference equation. Then, to compute the value of the analytic continuation along the q-spiral $\lambda q^{\mathbb{Z}}$, it is sufficient to use recursively the q-difference equation.

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