Normal forms of topologically quasi-homogeneous foliation on $(\mathbb{C}^2, 0)$

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Abstract

Our aim is to provide a formal form for class of topologically quasi-homogeneous foliations. It includes three parts: the principle term, the hamiltonian term and the radial term. This is a generalization of the results obtained in [2] for the topologically homogeneous class.

Notations and definitions

A germ of holomorphic function $f : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ is *quasi-homogeneous* if f belongs to its jacobian ideal $J(f) = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$. Then, there exist coordinates (x, y) and positive coprime integers k, ℓ such that $R(f) = d \cdot f$, where $R = kx\frac{\partial}{\partial x} + \ell y\frac{\partial}{\partial y}$ is the quasi-radial vector field [4]. In these coordinates, f can be written as

$$f = cx^{n_0}y^{n_\infty}(y^k - a_1x^\ell)^{n_1} \cdots (y^k - a_bx^\ell)^{n_b},$$

where the multiplicities satisfy $n_0 \geq 0, n_\infty \geq 0, n_b > 0$.

A germ of holomorphic foliation $\mathcal{F}$ is called *topologically quasi-homogeneous* if $\mathcal{F}$ is non dicritical without saddle-node singularities and the separatricies of $\mathcal{F}$ are topologically conjugated to the zero level set of a quasi-homogeneous function.

Let $\mathcal{Q}(f)$ be the class of holomorphic germs of 1-forms that define topologically quasi-homogeneous foliations having separatricies topologically conjugated to $\{f = 0\}$. Two germs ω, ω' in $\mathcal{Q}(f)$ are called *orbitally equivalent* if there exist a local change of coordinates ϕ and an invertible function u such that

$$u.\phi^*\omega = \omega'.$$

Moreover, if $D_0\phi = \mathrm{Id}$ and $u(0) = 1$ then the equivalence will be called *strict*.

Denote by $\mathcal{Q}^d(f) = \{\omega_d = a_{d-k}dx + b_{d-\ell}dy : \omega_d \in \mathcal{Q}(f_0)\}$ where a_{d-k}, $b_{d-\ell}$ are polynomial functions of degree $d-k$ and $d-\ell$ respectively. We also denote

$$\mathcal{Q}(\omega_d) = \{\omega \in \mathcal{Q}^d(f) : \omega = \omega_d + \omega_{d+1} + \omega_{d+2} \dots\}.$$

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Main Theorem

For generic $\omega_d \in \mathcal{Q}^d(f)$, each germ $\omega \in \mathcal{Q}(\omega_d)$ is strictly formally orbitally equivalent to a unique form $\omega_{H,S}$ that is written

$$\omega_{H,S} = \omega_d + d\left(x^{\varepsilon_0}y^{\varepsilon_\infty}H\right) + S\left(\ell y dx - kxdy\right)$$

where H is a polynomial function

$$H(x,y) = \Sigma h_{ij}x^iy^j,\ ki + \ell j \geq k\ell n + 1,\ i \leq \ell n - 2,\ j \leq kn - 2,$$

and

$$S(x,y) = \sum_{j=0}^{kn+\varepsilon_\infty-2} s_j(x)y^j$$

is polynomial in the y variable of degree less than $kn + \varepsilon_\infty - 1$ whose coefficients, $s_j(x)$, are formal series on x.

- This formal form is the same as the one in [2] when we restrict to the topologically homogeneous case $(k = \ell = 1)$.
- The number of free parameters in the hamiltonian terms H is consistent with the dimension of the moduli space of topologically quasi-homogeneous functions computed in [1]

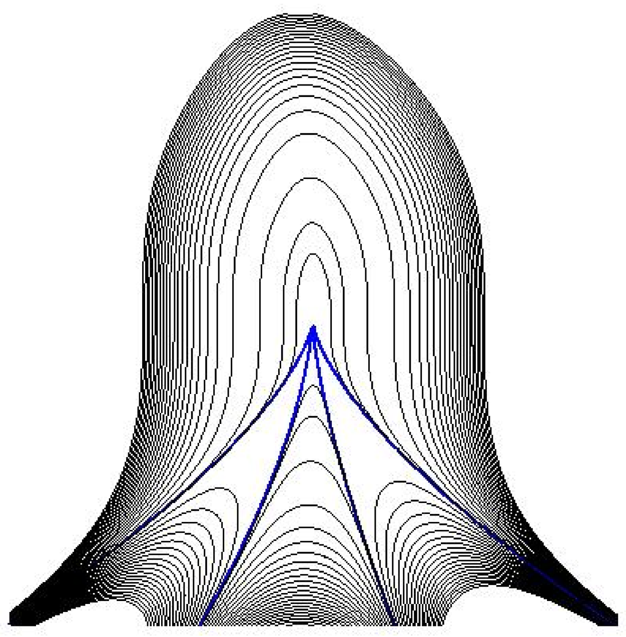

Figure: Topologically quasi-homogenous foliation

Generic conditions

- We require for $m = 1, \dots, k\ell n - k(1-\varepsilon_0) - \ell(1-\varepsilon_\infty) - 1$, $P_m(\omega_d) \neq 0$, where P_m are polynomial functions defined on the coefficients of ω_d. They appear when we compute the determinant of some system of linear equations.
- At least one of the two axes which correspond to $\{x^{\varepsilon_0} = 0\}$ and $\{y^{\varepsilon_\infty} = 0\}$ is invariant for ω_d and the Camacho-Sad index of this invariant curve is not a rational number.

Example

For $n = 2$, $k = 3$, $\ell = 2$, $\varepsilon_0 = \varepsilon_\infty = 1$, then the formal forms are of the 1-forms

$$a_{14}dx + b_{15}dy + d(x^3y^5) + S(\ell y dx - kxdy)$$

where $S(x,y) = s_0(x) + s_1(x)y + \dots + s_5(x)y^5$.

Future work

To find the missing invariants in the problem of classification of germs of foliations. A direct consequence of the construction of the semi-universal equisingular unfolding in [3] is that the separatricies and the holonomies are not sufficient to classify in general. In [2], the authors show that two topologically homogeneous foliations are conjugated if they have the same holonomies and same hamiltonian parts after formalisation. But the proof is rather analytical and lacks of geometry. Trying to find a geometrical approach and understanding the geometrical contribution of the hamiltonian part may help us to describe the missing invariants.

References

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